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A Textbook Approach

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1.Introduction

In economics textbooks equilibrium analysis is usually introduced by the standard two-good model. Within this simple framework all the basic questions of equilibrium analysis - that is existence, uniqueness and stability of a market clearing price vector - can be thoroughly discussed with comparatively simple analytical means. The same applies to the pitfalls of this theoretical approach, all of which can be addressed within the two-good model without being too demanding in terms of advanced mathemathics.
This note is concerned with a pitfall of equilibrium economics which had been perceived as such no sooner than the early 1980s as 'chaos' became known to economists. Surprisingly, so far very little attention has been paid to this new threat to mainstream economics by equilibrium economists, let alone by textbook authors.

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To be more specific, the point in question is the occurance of chaos in standard equilibrium dynamics as it is introduced in textbooks for undergraduates. We will show in this note that chaos can affect the simplest version of a Walrasian exchange economy, the familiar two-good model.

The paper is organized as follows: We begin by introducing the basic assumptions of the standard general equilibrium model (i.e. Walrasian exchange economy). Then the assumptions which allow 'chaotic traps' in the two-good case are introduced. Finally we discuss the conditions under which chaos may occur.

2. The Standard General Equilibrium Model

In the language of dynamical systems theory, the basic assumptions of a Walrasian exchange economy read as follows

(1) The state space is the set of all ($\sum p_i^2 = 1$, i=1,2,3,...,k) - normalized prices, that is, the positive orthant of the unit sphere $S_+^{k-1} = \{p \text{ in } D^k : || p || = 1, p \geq 0\}$, D^k the unit disk

(2) z(p) represents the k-vector of excess demands, z(p) homogeneous of degree 0 in prices

(3) $z(p): S_+^{k-1} \to R^k$, z(p) C^r, $r \geq 1$

(4) $p \cdot z(p) = 0$ for p in S_+^{k-1} (Walras' Law)

(5) $z_i(p) > 0$, if $p_i = 0$, i=1,2,3,...,k (Desirability)

Assumption (1) defines the price space of the equilibrium model so that it becomes a differentiable manifold with the mathematically nice properties of a sphere. Assumption (2) excludes money illusion. Assumption (3) makes sure that the excess demand function z(p) meets certain differentiability requirements. Assumption (4) expresses Walras' Law, which guarantees that z(p) lies in the tangent space of the sphere S_+^{k-1}.

That is to say, z(p) is a vector field on S_{+}^{K-1}. Assumption (5) excludes not only boundary equilibria but also allows the application of an elegant index argument to prove that our model has at least one fixpoint or equilibrium (see Varian (1984)).

At the heart of the standard equilibrium model is the excess demand function z(p). This function and its properties are elemental for finding answers to the three basic questions of equilibrium analysis: Is there an equilibrium price? Is there uniqueness? Is there a natural adjustment process leading the economy from a non-equilibrium price towards an equilibrium? (Hildebrand-Kirman (1988)). On the basis of the assumptions made in general equilibrium theory, only the existence problem has so far been resolved satisfactorily. The uniqueness and stability question remains still open and there is no hope that there will ever be an analytically satisfactory answer to both problems.

All we can hope for is, in general, local uniqueness. If we assume that the consumers have 'smooth ' indifference curves, then there is a finite number of price equilibria and, in general, this number will be odd (see, for example, Hildebrand-Kirman (1988) p.47).

Nothing, however, can be said about the stability properties of these equilibria. Under the given assumptions the excess demand function z(p) is rather arbitrary and with it the disequilibrium adjustment process, say, of type Walrasian tatonnement (see Debreu (1974)). As Scarf (1960) has shown, with three goods, there can be a stable limit cycle if prices adjust according to Walras' tatonnement. With four goods, the tatonnement process can even generate chaotic dynamics (see Kehoe (1988)). Only in the special two-good case things seem to look a bit better. If we do assume that the disequilibrium adjustment process be of type Walrasian tatonnement (i.e. the price of a good increases if the excess demand for it is positive et vice versa) then, in the simple two-good world, as attractor (repellor) there are only equilibria, at least one of which is locally stable (Hildebrand-Kirman (1988) p.48).

But even the relatively nice dynamics in the two-good world of equilibrium economics falls when we drop the assumption which guarantees it. We will see that when assumption (5) is replaced by an equally plausible, or better, by an equally implausible other technical assumption, the familiar two-good textbook model behaves dynamically as strange as the n-good model. That is to say, cycle and chaos are no longer excluded from the simple two-good world of equilibrium economics.

3. Equilibrium Dynamics: A Slightly Different Two-Good World

Equilibrium economists are used to ruling out the possibility of equilibria at a zero price by making some sort of desirability assumption (Varian (1984) p. 245). In general, assumptions similar to (5) are chosen so that $z_i(p) > 0$ when $p_i = 0$. But desirability conditions are primarily technically motivated rather than theoretically. The exclusion of boundary equilibria makes it, in general, mathematically much handier to prove that there exist equilibria for $p > 0$. For our purpose, however, it is mathematically more convenient to allow a special boundary equilibrium. We say that a market is to be taken in equilibrium when the market price is zero (i.e., the paradise case). That is to say, $z(p)=0$ when $p=0$. In order to make this assumption theoretically more acceptable, we further state that this boundary equilibrium be globally instable. For the sake of simplicity, besides this boundary condition we demand global uniqueness. That is to say, there is only one equilibrium for $p > 0$ allowed. As for the stability property of this equilibrium, we only demand that local stability be possible when prices adjust locally according to the Walrasian tatonnement process or the so-called law of supply and demand.

These requirements are fulfilled when the economy behaves as follows: For slightly positive prices the desirability assumption comes into play and $z(p)$ increases. If p exceeds a certain threshold $p' > 0$, $z(p)$ turns around and decreases for all $p > p'$. That is to say, all goods are substitutes for prices $p > p' > 0$.

Since we are dealing with the two-good model, let $z(p)$ be the excess demand function for good 1 and p the price of good 1, $0 < p < 1$. Because of Walras' Law, the excess demand function for good 2 needs, of course, no explicit consideration.

One candidate for an excess demand function which meets our requirements is the following map

$$(1^*) \qquad z(p) = Kp(1-p)-p$$

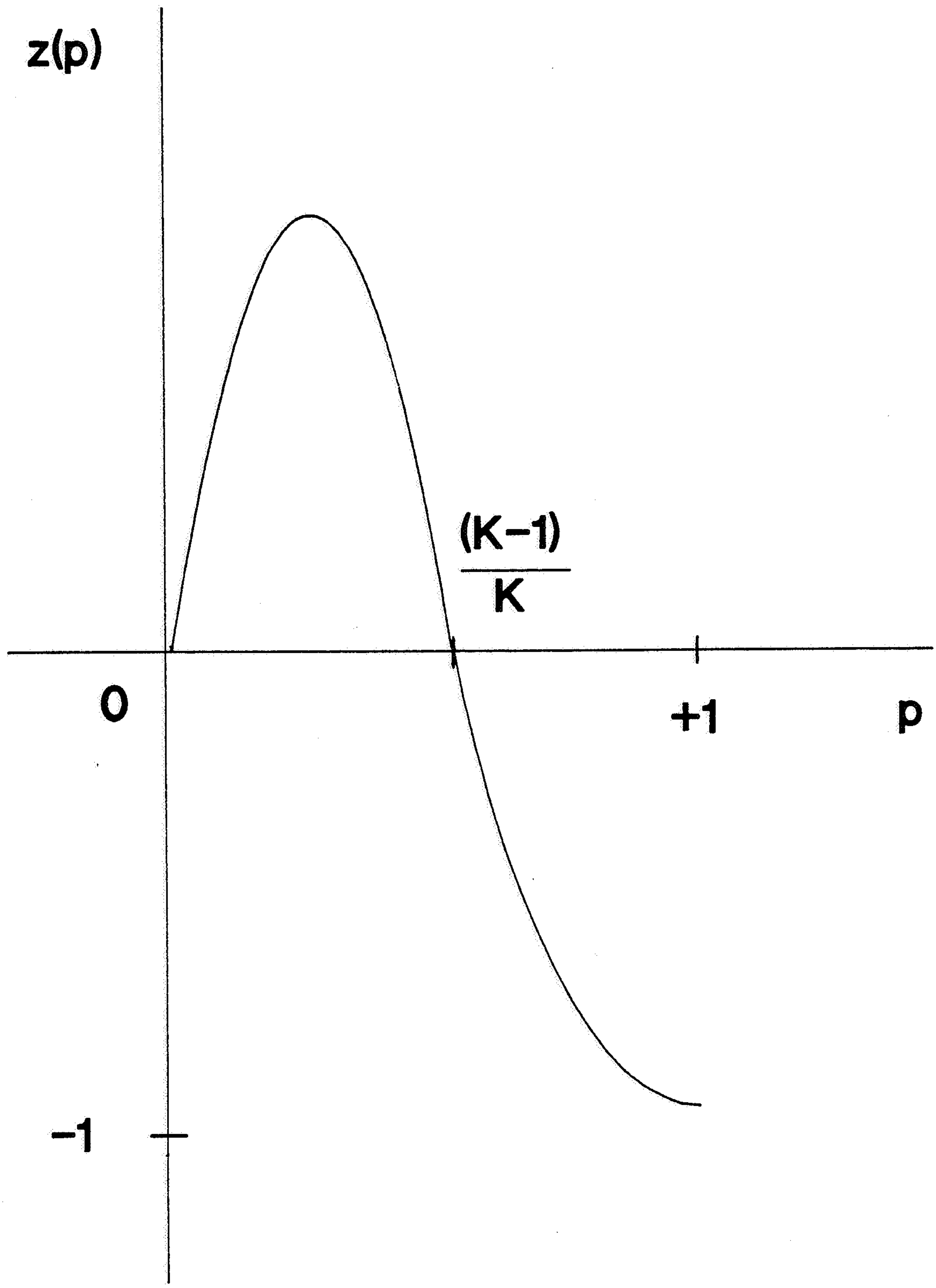
$$\text{where } K \in [1,4] \subset R$$

The excess demand function (1*) goes from zero to minus one, $p \in [0,1]$, crossing the p-axis at $(K-1)/K$ from above (Fig.1). In other words, we get two equilibria, that is, $z(p) = 0$ for $p = 0$ and $p = (K-1)/K$, respectively. Since $1 \leqslant K \leqslant 4$, the equilibrium prices stay within the interval $[0,1]$.

The following discrete adjustment rule also meets our requirement to permit a Walrasian tatonnement process in a local neighborhood of $p=(K-1)/K$

$$(2^*) \qquad p(t+1)-p(t) = z(p)$$

Fig. 1



Substituting (1*) into this difference equation of first order translates into the following mapping

$$(3^*) \qquad p(t+1) = Kp(t)(1-p(t))$$

which maps [0,1] onto itself for $1 \leqslant K \leqslant 4$. Equation (3*) may look quite familiar to the mathematically inclined reader, it is the famous logistic map.

4. The 'Chaotic Trap'

No later than now should it become clear why we have chosen an excess demand specification and an adjustment rule which lead to the logistic mapping. The logistic function belongs to the family of quadratic maps whose qualitative properties have been most thoroughly studied by 'chaos mathematicians' (see, e.g. Devaney (1986)). In addition, it can easily be shown that difference equation (3*) satisfies all stability requirements we stated in the previous chapter. For example, for $K \in [1,3)$, we have an equilibrium structure only, with $p=0$ unstable and $p=(K-1)/K$ stable (for $K \in [1,2)$ monotonic convergence (Fig. 2) and for $K \in [2,3)$ oscillation (Fig. 3)).

For $K \in [3.57,4]$, however, we get a dynamic configuration with seemingly erratic (chaotic) trajectories leading away from the two equilibria $p=0$ and $p=(K-1)/K$, respectively, but still staying within the state space of our model (Fig. 4).

In order to determine if the dynamic behavior of our model for $K \in [3.57,4]$ is in fact chaotic we need to understand what is meant by chaos. Loosely speaking, chaos occurs when for a typical initial value there is both sensitive dependence on initial conditions and aperiodic motion (Kelsey (1988) p. 9).

Fig.2

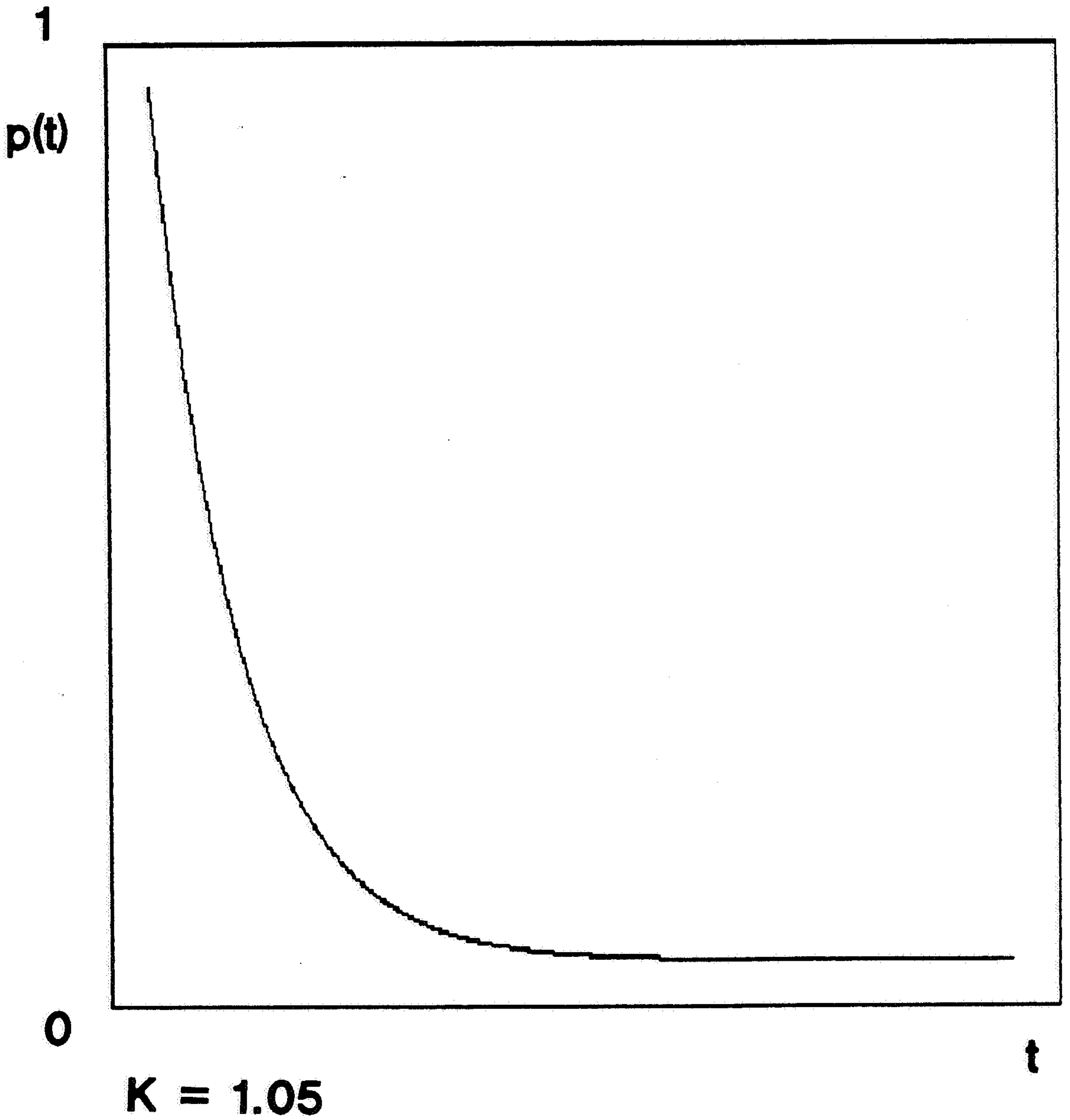


Fig.3

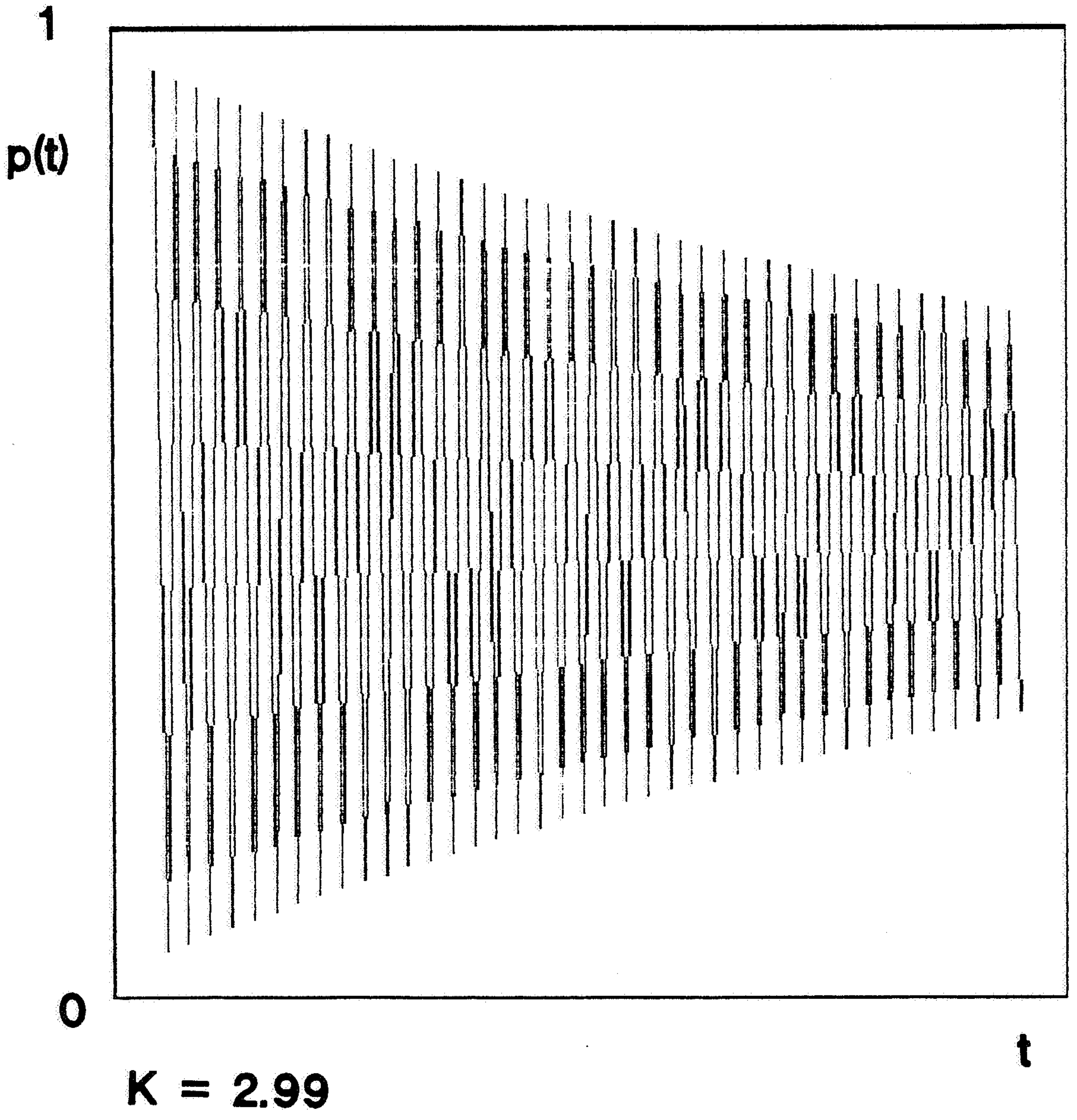
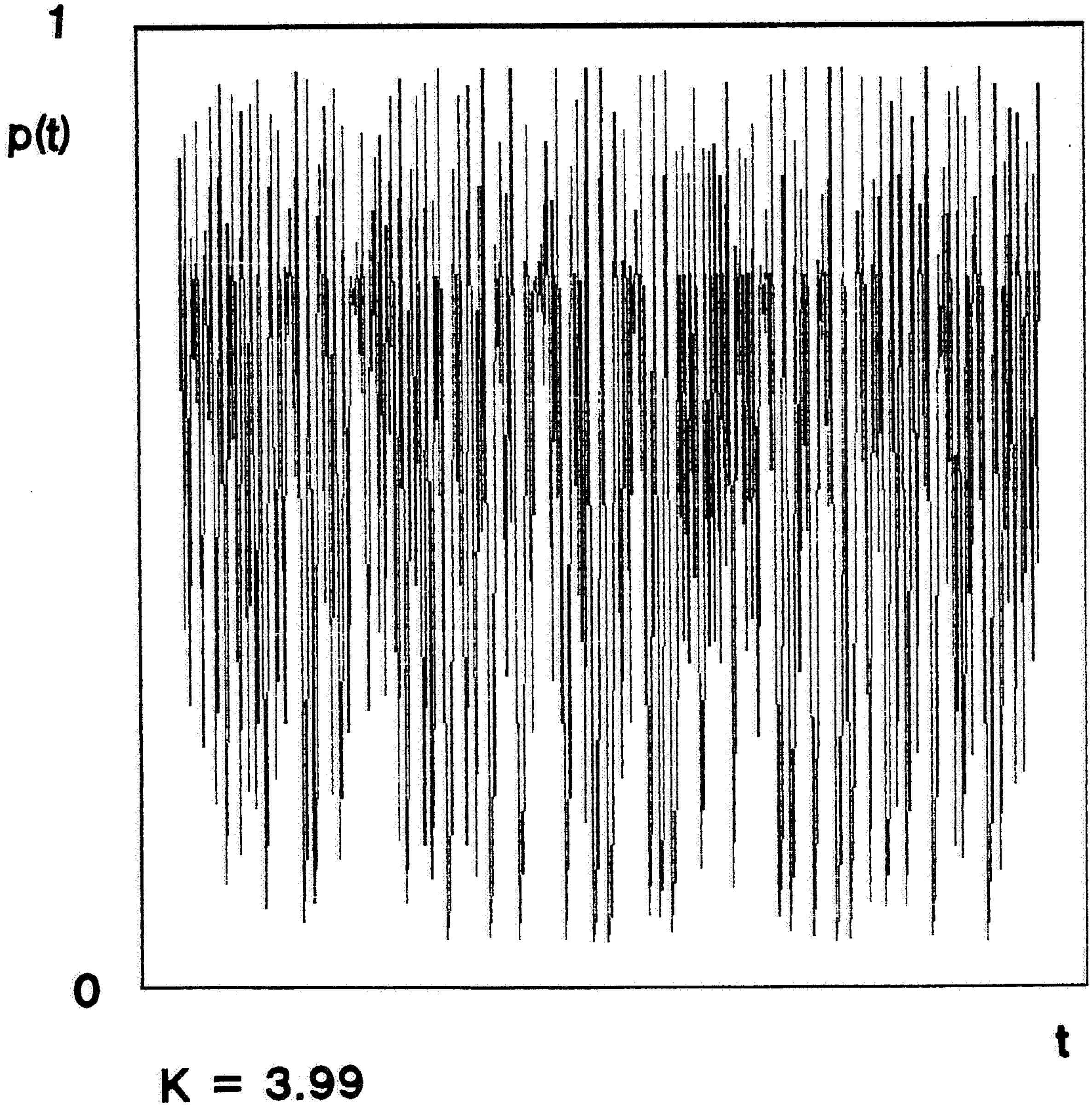


Fig.4



For our analytical purpose, we need a more formal definition. In doing so we follow M. Woodford's version of strong chaos. According to Woodford (1989) p. 320, a function $x(t+1)=f(x(t),K)$ is strongly chaotic if

(i) it has at most a countable number of periodic points and all of its points are unstable, and

(ii) for almost all initial conditions $x(0)$ in X, the set of initial conditions, there exists a probability measure u on X that is absolutely continuous with respect to Lebesque measure and that describes the asymptotic frequency distribution for $x(t)$. What is meant by that is that chaotic trajectories should be 'observable' for almost all initial conditions.

Jacobson (1981) proved that the logistic function $x(t+1)= Kx(t)(1-x(t))$ is strongly chaotic for a set of values of K between 3.57 and 4 that is of positive measure.

(For the sake of completeness, for $K \in [3,3.57)$ we get an equilibrium-periodic orbit configuration where the two equilibria $p=0$ and $p=(K-1)/K$ are unstable and the cycle is asymptotically stable).

5. Concluding remarks

What does it mean from the viewpoint of equilibrium dynamics when our simple two-good model governed by a Walrasian adjustment process starts with $K \in [3.57,4]$, $p(t=0)$ an arbitrary non-equilibrium price? It simply means that the auctioneer 'whose sole function is to search for the market clearing prices' (Varian (1984) p.244) fails to find an equilibrium regardless of however close to one of the two equilibria the starting point is chosen and regardless of the fact that in the course of the Walrasian tatonnement a price offer is never being made twice. That is to say, even though the Walrasian auctioneer never stops searching for new price offers he will not find the market clearing prices he is so desperately looking for. A rather weird economy which indeed deserves to be called chaotic. To make matters worse, similar results can be obtained for a rather wide class of functions of type (3*). If a family of functions of type $p(t+1)=f(p(t),K)$ is single-peaked, satisfies a special convexity condition (i.e., f should have negative Schwartzian derivatives) and is increasing in K, then the qualitative properties are similiar to those one gets for the logistic map (for details, see Devaney (1986)).

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