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ON STOCHASTICS AND NON-LINEARITY IN MACROECONOMICS - SOME COMMENTS

Franz R. Hahn

1. Introduction

The business cycle model of N. Kaldor published in 1941 is one of the few paradigmatic models in modern macroeconomics. Despite its advanced age Kaldor's model remains very challenging, especially from an analytical point of view. The attraction of this model comes solely from its non-linearity and, as a result, from its seemingly inexhaustible possibilities of analytically relevant extensions.

In the following study an attempt is made to discuss some basic aspects of the interplay of randomness and non-linearity within Kaldor's famous business cycle model in H. Varian's interpretation as a simple elementary catastrophe model.

The note proceeds in detail as follows: In the second section a cursory description of the original model as well as the ex - tended version of H. Varian is given. The third section introduces a simple stochastic extension of the Kaldor-Varian-model (hereafter abbreviated as K-V-model). In the fourth section the admissibility of linear approximation within a basically non-linear model is thoroughly discussed. Finally, an attempt is made to solve the so-called First Passage Time Problem for the underlying stochastic catastrophe model.

2. The K-V-Model

The original business cycle model of N. Kaldor can be presented by the following two non-linear differential equations (see Chang/ Smyth (2))

$$(1) \qquad \dot{y} = \alpha\left[c(y) + i(y,k) - y\right] \quad , \alpha > 0$$

$$(2) \qquad \dot{k} = i(y,k) - k_0$$

$$(3) \qquad \dot{y} =: \frac{dy}{dt} \quad , \quad \dot{k} =: \frac{dk}{dt}$$

where y equals real income, k represents the fixed capital stock, k_0 the autonomous depreciation and t stands for time.
The parameter $\alpha > 0$ represents the adjustment speed. The consumption function represented by c(y) follows the standard convention $0<c_y<1$.
As for the investment function i(y,k), Kaldor established a non-linear relationship between investment i and income y.
The non-linear investment function was specified in detail as follows:

$$(4) \qquad i_y > 0 \ , \quad i_{yy} > 0 \text{ for } y < \bar{y} \ , \ i_{yy} < 0 \text{ for } y > \bar{y}$$

$$(5) \qquad i_k < 0$$

$$(6) \qquad i_{yk} = i_{ky} = 0$$

Kaldor justified these unfamiliar specifications only by plausibility arguments failing to give his proposal a more theoretically based meaning. Instead he resorted to the heuristically based argument that a low level of economic activity would in practice often lead to a small marginal propensity to invest, by reason of existing excess capacities. On the other hand, during boom phases, so he argued, the marginal propensity to invest is most probably negatively affected by high interest rates and high real wages.[1)]
It can be shown that the $\dot{y}=0$ - curve in the (y,k) - plane is strictly monotonically increasing for $y \in (y_a, y_b)$, otherwise strictly monotonically decreasing (see Figure 1).

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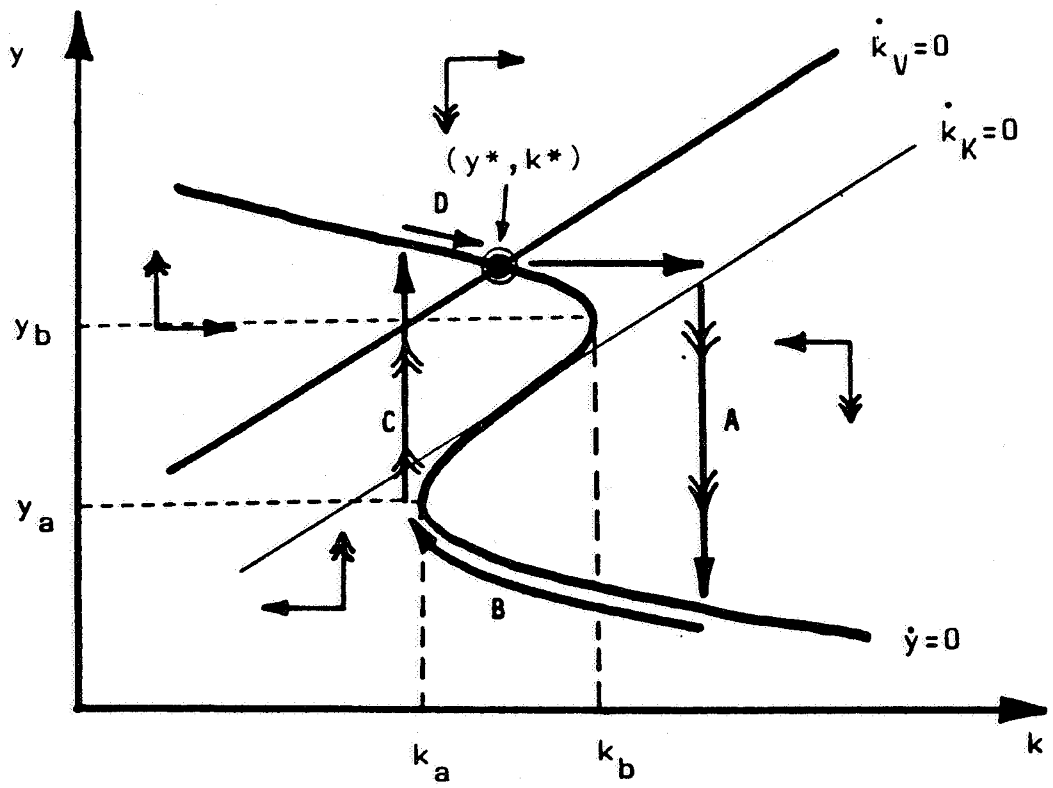
1) See Kaldor(16), p. 81

The upper and the lower leaf of the $\dot{y}=0$ - curve represents geometrically the set of locally dynamically stable points. The folded middle piece of the $\dot{y}=0$ - curve is the geometrical region of the locally dynamically unstable points of equilibrium.

The $\dot{k}=0$ - curve, however, is strictly monotonically increasing over the whole domain.
Chang/Smyth (2) proved that under the assumptions made above only one common fixpoint exists. That means geometrically, the curves $\dot{y}=0$ and $\dot{k}=0$ cross only once.
In his famous article Kaldor investigated only the qualitative behaviour of that dynamical model where the common equilibrium is dynamically unstable. In other words, he studied the special case in which the $\dot{k}=0$ - curve crosses the $\dot{y}=0$ - curve in its monotonically increasing region (see Figure 1).
Contrary to N. Kaldor, H. Varian (25) analyzed the qualitative behaviour of Kaldor's model by assuming that the only equilibrium point is dynamically stable and, secondly, is located on the upper leaf of the $\dot{y}=0$ - curve.
In geometrical terms this means that the $\dot{y}=0$ - curve is crossed by the $\dot{k}=0$ - curve on its upper leaf (see Figure 1).
Additionally, Varian suggested that this equilibrium can be said to be a long run equilibrium or a long run equilibrium growth path which is locally stable but globally unstable.
It is therefore reasonable to suppose that the equilibrium (y^*,k^*) is placed within the bimodal region.
By interpreting the variable y as a so-called fast variable, ensured by a very large adjustment coefficient $\alpha>>0$, and the variable k as a so-called slow variable or parameter, the short term equilibrium configuration can be construed as a double fold in the sense of the elementary catastrophe theory (in this case the catastrophe set consists of k_a and k_b).

FIGURE 1



At the parameter values k_a and k_b the state variable y suffers a discontinuous change that is commonly called an elementary catastrophe in terms of the mathematical bifurcation theory.
Macroeconomically seen, the K-V-model represents a simple formal description of Leijonhufvud's plausible hypothesis of the so-called corridor stability combined with the overinvestment hypothesis of the traditional theory of business cycle.
Beyond that Varian extended this model by taking a wealth variable into consideration which can be interpreted as a splitting factor in terms of the elementary catastrophe theory.
The thus extended model can be mathematically treated similar to a cusp catastrophe model by which the different aspects of economic recessions and depressions can be shown in a very elegant manner.
Concerning the stochastic extension of the K-V-model in the next section, we confine our interest, however, to the fold model on grounds of a simpler geometrical presentation, not least because all conclusions drawn for the fold remain true for the cusp.

3. A Simple Stochastic Version Of The K-V-Model

The interpretation of Kaldor's model as a fold catastrophe by H. Varian (25) permits us to introduce two simplifications which are qualified to facilitate the following formal investigation.
First, the dynamical system (1) and (2) can be formally treated like a so-called gradient system and, second, under the underlying assumptions it is allowed to confine the formal investigation to the canonical fold only.[2)]
These simplifications leave the K-V-model qualitatively unchanged, but they lead to a significant decrease of formal efforts regarding the analysis of the equilibrium configuration of the model with respect to continuous changes of the slow variable k.
What is left to be done mathematically is simply to analyze the singularity structure of a fourth-degree polynomial or, equally, the stationary points of an appropriate canonical gradient system (canonically in terms of the elementary catastrophe theory).

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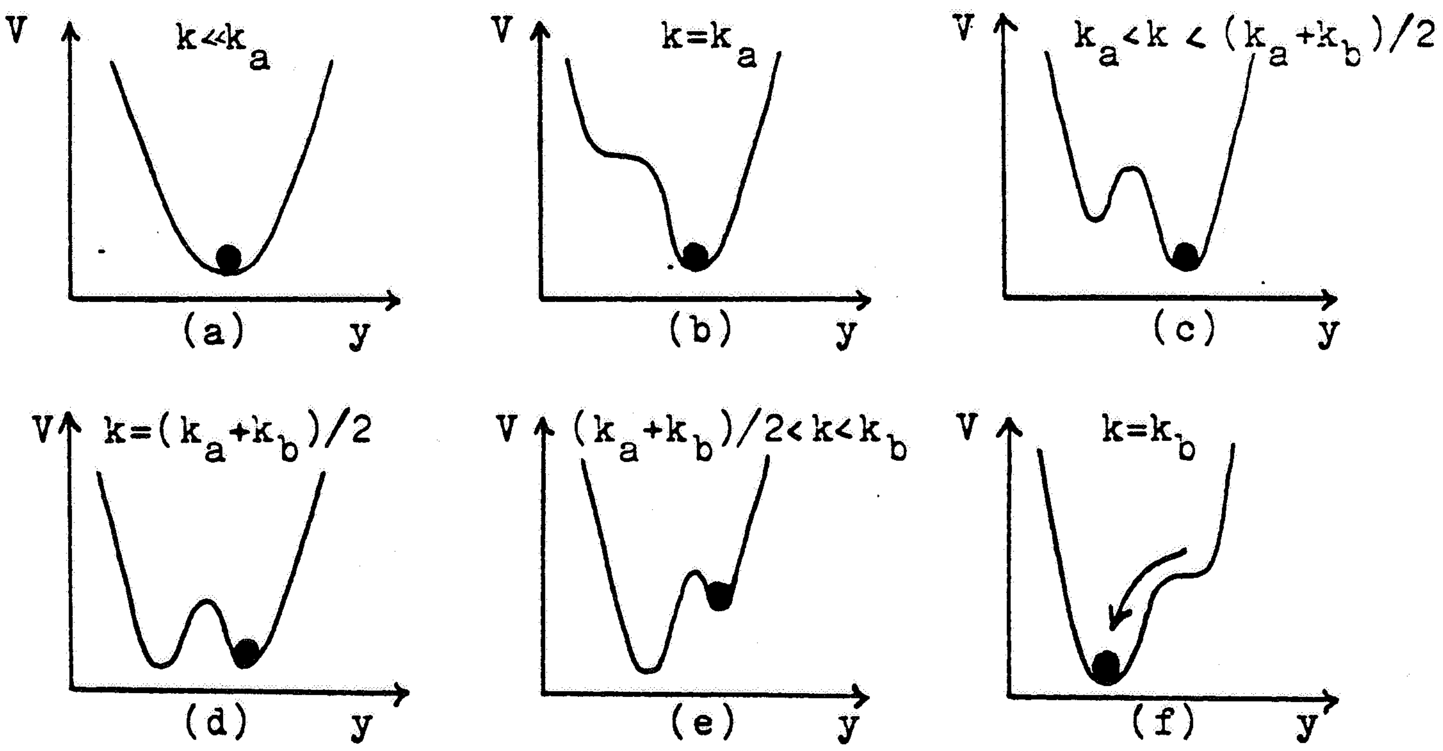
2) A thorough description of the gradient system is given in Smale/Hirsch (14).

FIGURE 2



That means, in other words, that the most important qualitative aspects of the stochastic K-V-model can be analyzed on the basis of the following non-linear, canonical differential equation, which shows all properties of an appropriate canonical gradient system being necessary for the formal investigation of a fold catastrophe.

$$(1') \qquad \dot{y} = -ky + ay^2 - by^3$$

$k,a,b > 0$, parameter

$y(0) = y_0$, initial condition

The set of stationary points of this non-linear differential equation with respect to the parameter $k \in (-\infty,+\infty)$, still representing the fixed capital stock, is topologically equivalent to the $\dot{y}=0$ - curve of Figure 1 by the constancy of the two parameters $a,b>0$ (the slow variable k still remains determined by equation (2)).
The equation (1') can be considered in this context as a kind of phenomenological equation.
The phenomenological potential, correlated with the relatively simple gradient system (1') can be written as follows

$$(2') \qquad V(y,k) = \frac{k}{2}y^2 - \frac{a}{3}y^3 + \frac{b}{4}y^4$$

Some of the potential's graphs typical for the K-V-model are shown in Figure 2.
We assume now that only the fast variable y is affected by random disturbances or random shocks. The slow variable k, however, is assumed to be but slightly disturbed by random shocks. Thus, we can neglect further on the exogenous stochastic influence on the slow variable k.

This additional simplification permits us to conduct the stochastization of the K-V-model at the reduced dynamical system (1') instead of the original model.
The simplest and most common way to randomize an equation like this is to add a stochastic term which obeys the classical conditions of the Gaussian white-noise process.
By doing so, we obtain the following non-linear differential equation with a random inhomogeneous part which follows a white noise

$$(3') \qquad dy = (-by^3 + ay^2 - ky)dt + db(t)$$

$$(4') \qquad E[dB(t)] = 0$$

$$(5') \qquad E[dB(t)^2] = Q \cdot dt \quad , \quad Q \in \mathbb{R} \text{ and fix}$$

$$\text{whereby} \quad \frac{dB(t)}{dt} = W(t) \quad , \quad t \geqslant 0$$

$W(t)$ - Gaussian white noise

The equation (3') is a simple representative of a special class of stochastic differential equations which play particularly important roles in the control theory and in the communication theory. This class of stochastic differential equation is known as Ito - equation.
The Ito - equation (3') describes a special stochastic process, called Wiener or Brown process.
The Wiener process is commonly defined by a family of random variables $\{X(t)\}$, $0<t<\infty$, which obey the following conditions: (a) $X(0)=0$; (b) the increments $X(s_i+t_i) - X(s_i)$ are stochastically independent for every finite set of intervals (s_i, s_i+t_i); (c) $X(s+t) - X(s)$ is normally distributed for all $s \geqslant 0, t \geqslant 0$.[3)]

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3) See Fellner (6), Vol. II, p. 99

In contrast to the deterministic case we are no longer interested in single equilibrium trajectories of the Ito - equation, but rather in the density of all trajectories and its change with respect to the parameter k.
With the aim of solving this problem we use the following theorem originating in the theory of Ito - equation

Theorem: Let $f(y,t/y_0)$ be the probability density of the random variable y at the time $t>0$, given the initial condition $y_0=0$, $t_0=0$.

Given the Ito - equation

$$(1*) \quad dy(t) = g(y(t),t)dt + dB(t)$$

with the initial condition $y(0)=y_0$, then there exists the following deterministic partial differential equation for the transition density

$$(2*) \quad \frac{\partial f(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[g(y(t),t)f(y,t)\right] + \frac{1}{2} Q \frac{\partial^2}{\partial y^2} f(y,t)$$

The equation (2*) is known as Fokker-Planck-equation.

A proof of this theorem is given in Soong (23), chapter 7.

The Fokker-Planck-equation for the random differential equation (3') - formally a special forward-equation - can be written in the following form

$$(6') \quad \frac{\partial f(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[(-by^3+ay^2-ky)f(y,t)\right] + \frac{1}{2}\cdot Q\cdot\frac{\partial}{\partial y^2} f(y,t)$$

To facilitate the formal investigation we assume that the transition time $t-t_0$ approaches infinity. That enables us to focus our formal attention entirely towards the determination of the stationary solution of the Fokker-Planck-equation or the steady-state distribution, denoted by $f_s(y,k)$.

We obtain the stationary probability density for the Fokker-Planck-equation (6') simply by setting the term $\frac{\partial f}{\partial t}$ equal to zero and solving the remaining ordinary differential equation by means of elementary methods.
The steady-state distribution in this case takes then the following form

$$(7') \qquad f_s(y,k) = \frac{c}{Q}\cdot \exp\left\{\frac{2}{Q}\cdot\int_0^y(-bs^3+as^2-ks)ds\right\}$$

whereby $f_s(y,k)$ reasonably satisfies following boundary conditions

$$(8') \qquad f_s(\pm\infty) = 0$$

$$(9') \qquad \frac{df_s(y,k)}{dy}\underbrace{(\pm\infty)}_{y} = 0$$

Hence, the stationary solution of the Fokker-Planck-equation (6'), upon substituting the term $\dot{y} = -DV(y,k)$ into equation (7'), can be finally written in the form

$$(10') \qquad f_s(y,k) = N\cdot\exp\left\{-\frac{2}{Q}V(y,k)\right\}$$

The term $N = \frac{c}{Q}$ represents the normalization constant that guarantees the following convention of the probability theory

$$(11') \qquad \int_{-\infty}^{+\infty} f_s(y,k)dy = 1$$

It can be easily seen from the equation (10') that the deterministic potential function $V(y,k)$ and the steady-state distribution $f_s(y,k)$ are related in an inverse manner. The steady-state distribution equals geometrically the inverted potential function, slightly distorted by the diffusion coefficient Q and the normalization constant N.

FIGURE 3

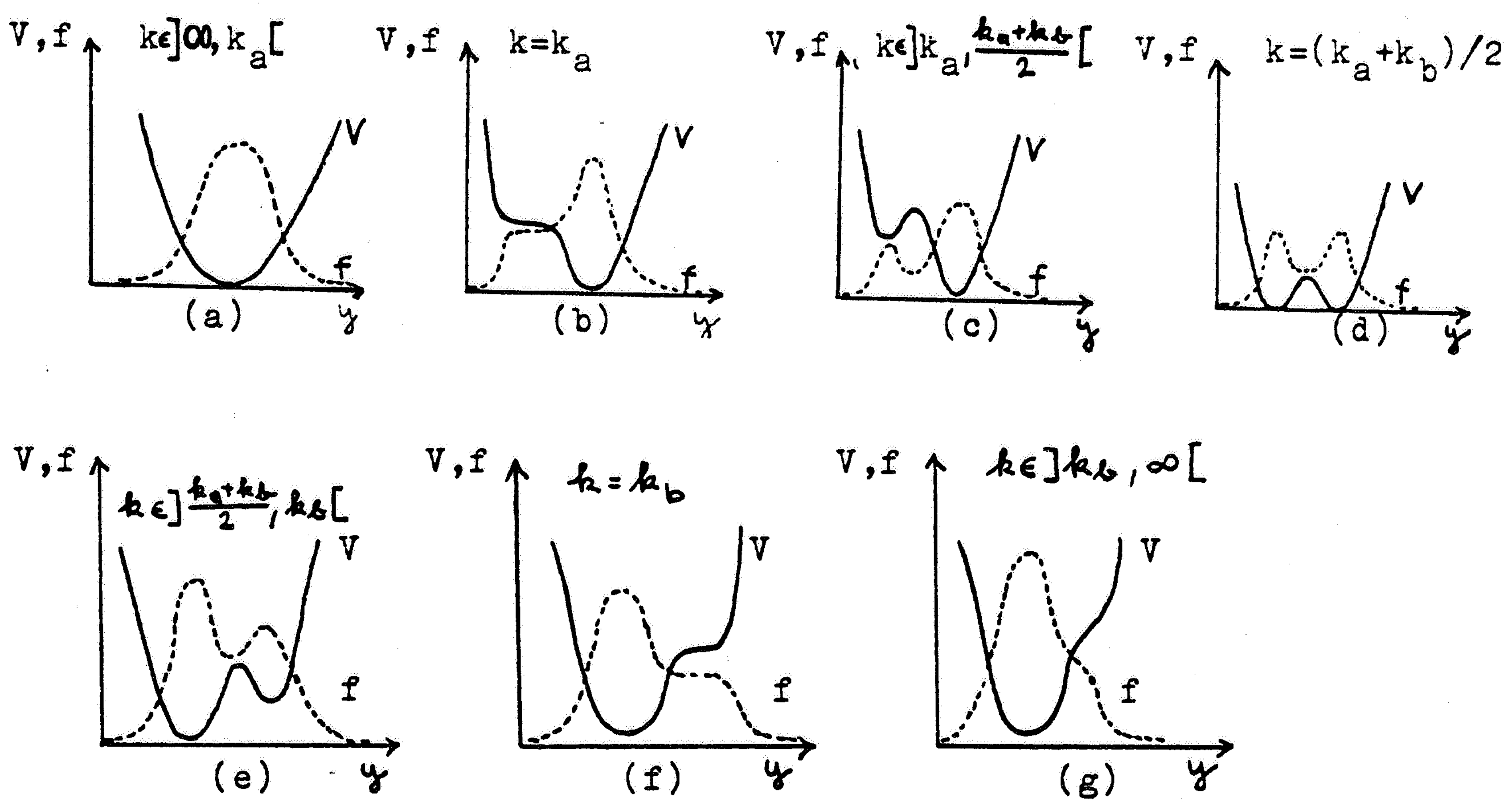
V, f
k∈]0, k_a[
V
f
(a)
y
V, f
k=k_a
V
f
(b)
y
V, f
k∈]k_a, (k_a+k_b)/2[
V
f
(c)
y
V, f
k=(k_a+k_b)/2
V
f
(d)
y
V, f
k∈](k_a+k_b)/2, k_b[
V
f
(e)
y
V, f
k=k_b
V
f
(f)
y
V, f
k∈]k_b, ∞[
V
f
(g)
y

Figure 3 shows the main difference between the stochastic and the deterministic K-V-model. The deterministic K-V-model is governed by the so-called Delay-Convention. The opposite is the so-called Maxwell-Convention. Following the latter rule the dynamical system tends towards the global minimum of the potential.
As for the stochastic K-V-model, particularly under these stochastic assumptions made above, the delay-rule is no longer exclusively valid, but neither is the Maxwell-rule. Between these opposing extremes lies the Fokker-Planck-equation.
Which convention under which conditions outweighs the other, will be discussed together with the First Passage Time Problem in the next section.
In contrast with the deterministic case, however, the findings of this section result in the impossibility of determining a bifurcation set for the randomized model. The catastrophe set in the stochastic case is simply degenerated to a fuzzy set. Catastrophes in the randomized context may no longer only occur at the parameter values k_a and k_b, qualitative changes can now be triggered off by every $k \in [k_a, k_b]$. The likelihood of the occurrance of such catastrophes depends on the actual size of the diffusion coefficient Q. Related to its qualitative status the randomized model is obviously more exposed to equally intensive shocks within this bimodal region than outside this region (beyond that, the probability density function in Figure 3 reflects very clearly the dilemma of the economic forecasters, whose predictions are in most cases based on at least bimodal densities).
Hence, this stochastic version of the K-V-model may be interpreted as a first simple attempt to combine Leijonhufvud's concept of the corridor stability and the traditional overinvestment hypothesis with the hypothesis of erratic shocks within one model.

4. On The Linearization Of The Stochastic K-V-Model

One of the main questions in macroeconomic model building is to what extent the so-called linearization can be theoretically justified. What is commonly meant by linearization is to approximate complex, non-linear relations and subjects by linear functions. Linear modeling is very common in macroeconomics for formal reasons only. Nevertheless, linearization as the simplest method of approximation is often tentatively based on the argument that all complex, non-linear relations can be sufficiently approximated locally by linear mappings. Linear approximations are said to possess locally the same qualitative properties as the subject being approximated by them.

That this is not valid in this general sense can be shown very clearly with the help of the simply randomized K-V-model.

We show in the following that just as in the local neighbourhood of catastrophe points, which represent the turning points in this cycle model, linearization will formally break down.

Presupposed as known from the stability theory, the qualitative behaviour of non-linear, dynamical systems close to locally dynamically stable fixpoints can be approximated by linear differential equations. That is essentially valid for random dynamical systems too, especially, if their stochastic terms obey a Gaussian white noise.

For the case in question this means that the gradient system (3') and its qualitative behaviour in the local neighbourhood of a locally dynamically stable equilibrium can be sufficiently reproduced by the following stochastic linear differential equation, if $k>0$

$$(1'') \quad \dot{y}(t) \approx -ky(t) + W(t)$$

where it is assumed that y is a small quantity close to the stationary state y_0 permitting us to neglect higher powers of (3').

For k<0, we have to establish some simple replacements.

Given

$$(2'') \quad y(t) = y_0 + \eta(t) \quad , \quad \eta \text{ very small}$$

the linearization of (3') can now be plainly written in the form

$$(3'') \quad \dot{\eta}(t) \approx -\underbrace{(2by_0^2 - ay_0)}_{\alpha > 0} \eta + W(t)$$

Even in equation (3''), terms of higher powers than 2 can be neglected because of the assumed smallness of η .

According to the Mean Square Theory of this type of random equations, the general solution of the random linear differential equation (3'') can be written as follows (see Soong (23), pp. 154)

$$(4'') \quad \eta = \exp\left\{-\alpha t\right\}\eta_0 + \int_0^t \exp\left\{-\alpha(t-s)\right\} W(s)ds$$

The first term on the right hand sight is negligible for very large t, so we can replace the general solution (4'') by the following approximation

$$(5'') \quad \eta = \int_0^t \exp\left\{-(2by_0^2-ay_0)(t-s)\right\} W(s)ds$$

The problem of approximating a basically non-linear random process by local linearization can now be summarized briefly as follows: Linearization of a basically non-linear random process reflects the local behaviour of the model being approximated qualitatively correct as long as the random variable has a finite (and possibly small) variance. This is a necessary precondition to hold up the principle of locality.
The variance of the random variable η has now the following form [4)]

$$(6'') \quad E[\eta(t)\eta(t)] = \int_{-\infty}^{+\infty} Q\left(\int_0^t \exp\left\{-2\alpha(t-s)\right\} W(s)^2 ds^2\right)\underbrace{\delta(q-q')}_{\text{Dirac-function}}dq$$

$$= Q\left[\frac{1}{2\alpha} - \underbrace{\frac{1}{2\alpha}\exp\left\{-2\alpha t\right\}}_{\approx 0 \text{ for } t \gg 0}\right]$$

As t was assumed to be very large, the last term on the right can be neglected in the following. From (6'') it can be seen immediately that a finite variance for η is only defined for $\alpha \neq 0$.
If $\alpha \rightarrow 0$, the linearization leads to a breakdown.
The term $\alpha = 2by_0^2 - cy_0$ has now the following two roots

$$(7'') \quad y_{0,1} = 0 \; ; \quad y_{0,2} = \frac{a}{2b}$$

These roots correlate with the following parameter values

$$(8'') \quad k_a = 0 \; ; \quad k_b = \frac{a^2}{4b}$$

which equals, as expected, both catastrophe points of the K-V-model. This leads us to the result that in the local neighbourhood of these catastrophe points, linearization is no longer a proper means of local approximation. Close to these points where the most significant change of the qualiative behaviour of the model is most likely to take place, local linearization is utterly invalid.

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4) See Haken (12), pp. 80

The fluctuation of the random variable y becomes ever larger the nearer it approaches one of the catastrophe points.
When k reaches k_a or k_b, the variance of y ends finally in infinity.
This phenomenon is called "critical fluctuation" in the physics of phase transitions (see Haken (12)). This effect occurs only accompanied with artificial linearizations. In reality, the variance of the stochastic state variable is limited even in the close proximity of catastrophe points which is due to the non-linear term in equation (3').

5. On The First Passage Time Problem

In the final section we want to discuss the crucial question whether predictions on the basis of these types of non-linear, stochastic models are theoretically possible. Statements in the form of expectation values - the common procedure of main-stream-econometrics mainly oriented on linear functions - are no longer representative at least within the relevantly bimodal region.
The question about the possibilities of forecasting on the basis of the underlying model relates very closely to the question by which convention - by the delay or the Maxwell convention - the evolution of the dynamical system is predominantly governed.
We mentioned above that this depends essentially on two factors: (a) on the magnitude of the "noise", and (b) on the shape of the potential function or the potential jet.
We are going to try now to indicate conditions under which predictions can be made of similiar accuracy as those made on the basis of linear models. For this, we will use the narrow relationship between the diffusion process of type Fokker-Planck and the theory of random processes in a similiar way as the physicist R. Gilmore recently did.(see Gilmore (10)).

To my knowledge , Gilmore is the first to have conducted a comprehensive mathematical investigation on this issue by combining it very cleverly with the so-called First Passage Time (FPT) Problem and the Relaxation Time (RT) Problem respectively.
The FPT is the time-span in which the endogenous random variable is most likely to switch from one to another local minimum of the potential.
The RT is, however, the time-span the perturbed endogenous state variable, slightly imbalanced by shocks, needs to return to its original equilibrium position. We denote hereafter the relaxation time by T_1 and the first passage time by T_2.
To evaluate T_1 it will do to restrict the formal analysis to the neighbourhood of one of the potential's local minimum.
That means, we can suppress all terms of the potential function of powers higher than 2.
We assume further that the local minimum of the potential lies at y_0 and the endogenous random variable y is at the time t=0 within a local neighbourhood of y_0, but $y(t=0) \neq 0$.
These preconditions allow us to express the RT-problem as follows

$$(1''') \quad \lim_{t\to\infty} f(y,t=0) \approx N\cdot \exp\left\{-k(y_0+\Delta y(t))^2 \frac{1}{2Q}\right\} = N\cdot \exp\left\{k y_0^2 \frac{1}{2Q}\right\}$$

$$\text{whereby } \Delta y(t) = \Delta y \exp\{-kt\}$$

The RT obviously depends on the magnitude of the controll parameter k. That yields to the following definition of T_1

$$(2''') \quad T_1(y) = \frac{1}{k}$$

The relation between T_1 and the control parameter k in the local neighbourhood of y_0 is evident and needs no further explanation. The "steeper" the potential's shape around the local minimum y_0 is, the shorter, on the average, T_1 will be.

The evaluation of T_2 is comparatively much more difficult than that of T_1.

The point of the FPT-problem is, simply expressed, to evaluate the time-span needed on average by the random state variable y to reach one of the interval boundaries $[a^*,b^*] \in \mathbb{R}$, whereby y is assumed to be an inner point and the evolution of y is governed by a diffusion process, described above.
For the parameter T_2, following relations are suggested:

$T_2(y) > 0$ for $y \in (a^*,b^*)$, and $T_2(y) = 0$ for $y = a^*$ or $y = b^*$.

It can be shown that the underlying diffusion process is governed by the following second order ordinary differential equation[5)]

$$(3''') \qquad Q \cdot \frac{\partial^2 T_2}{\partial y^2} - \frac{\partial V(y,k)}{\partial y} \cdot \frac{\partial T_2}{\partial y} = -1$$

This differential equation has to fulfil the boundary condition $T_2(a^*) = T_2(b^*)$.
By introducing the integrating factor $\exp\left\{-V(y,k)\,\frac{1}{Q}\right\}$, the differential equation (3''') can be replaced by the following integral for $a^* < y < b^*$

$$(4''') \quad T_2(y)=\frac{c}{Q}\int_{a^*}^{y} \exp\left\{\frac{1}{Q}\,V(y',k)\right\} dy' + \frac{1}{Q}\int_{a^*}^{y} \exp\left\{\frac{1}{Q}V(y',k)\right\}\left(\int \exp\left\{\frac{1}{Q}V(y',k)\right\} dy'\right) dy'$$

5) See Fellner (6), and Gilmore (10)

Gilmore solved this somewhat complex integral with the help of Laplace's method for a potential function, which is topologically equivalent to ours.
If you choose the boundary condition $T_2(a^*)$, $T_2(b^*)$ in such a manner that the FPT-problem is to be solved only for the transition from a local (metastable) to its neighbouring minimum, $T_2(y)$ has the following approximate solution

$$(5''') \quad T_2(y) = \frac{2\pi}{\left| \frac{\partial^2 V(y_{min},k)}{\partial y^2} \cdot \frac{\partial^2 V(y_{max},k)}{\partial y^2} \right|^{\frac{1}{2}}} \exp\left\{\frac{1}{Q}(V(y_{max},k)-V(y_{min},k))\right\}$$

This result can be pictured in a very convincing manner.
$T_2(y)$ lasts all the longer the steeper is the curvature of the potential around the local minimum and the neighbouring local maximum. This is expressed by the second derivative of the potential $V(y,k)$ at the points y_{min} and y_{max}.
Furthermore, $T_2(y)$ is all the longer the larger is the relation between the absolute difference of $V(y_{min},k)$ and $V(y_{max},k)$ and the variance Q.
We are now in the position to name conditions under which the underlying non-linear stochastic system is governed predominantly by one of the two rules. In brief, given $T_1(y) \ll T_2(y)$, the system obviously tends towards the delay rule, otherwise the Maxwell rule prevails.
In other words, if a significant difference between $T_1(y)$ and $T_2(y)$ exists, reliable predictions in relation to the evolution of the state variable y are most likely to be possible on the basis of a non-linear, stochastic model of the K-V-type.
What is more, these findings show very clearly how vague the term stability, predominantly used in linear economics, actually is and how much its analytical meaning depends on the modeling context in which it is embedded.

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