Errors and Omissions in the Treatment of Backlogged Orders in Management Science Literature

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1. Aim of the Paper

Determining the optimal post-order (or post-production) inventory level (starting stock, y) in a dynamic context requires an assumption of what happens if eventually demand exceeds supply. We have learned long ago to distinguish between the case where the excess demand can be deferred into the next period at a certain cost (the backlog case - BL) and the alternative case where this excess demand is lost (the lost sales case - LS).

Literature on this issue is surveyed in chapter 2 and classified into four categories: (1) explicit neglect of backlogs, (2) implicit neglect of backlogs, (3) explicit and correct treatment of the BL case, (4) explicit but wrong (or at least loose) treatment of the BL case. Surprisingly the categories (1) and (3) do not dominate.

In chapter 3 we develop what we understand as the correct explicit treatment of (partial) backlogging using the standard dynamic programming procedure. To make things simple we use a linear cost, zero lead time model, with i.i.d. demand distribution and an infinite horizon.

1) The author is especially grateful to Arthur Veinott and John Hey for many discussions on inventory problems. Though this paper reaches results which differ from those presented by Veinott and Hey, I want to stress that only the excellent and much more general contributions of Veinott and Hey provided me with the tools to improve on their work in a rather small side aspect. I am grateful for comments received by Prof. Erich Loitlsberger, Prof. Gerhard Orosel, Prof. Erich Streissler, Prof. Peter Swoboda, Prof. Georg Winckler and Michael Wüger, though I did not follow their advice in all respects.

In chapter 4 we develop the implications of the correct treatment of backlogs for optimal production from the point of view of an industrialized economy supplying sophisticated products under considerable uncertainty. These implications are confronted with some empirical data in chapter 5.

2. Categorization of the Literature with Regard to Backlogging

A tiny but hopefully representative part of the immense literature on optimal inventory stock is surveyed in table 1. We tried to classify the articles and books into four groups depending on their treatment of inventories.

- The first group wants to study the LS case and explicitly assumes that backlogging is not possible (e.g. Karlin 1958, Zabel 1972). This may create unrealistic models from the empirical point of view, but is a methodologically correct procedure.

- The second group - which seems the largest - implicitly assumes the LS case. This is an omission and even a serious omission if the authors authoritatively pronounce their formula as an "important" and "well known" result of inventory theory, without confessing that it simply depends on the implicit LS assumption. The result that uncertainty "naturally" increases the optimal inventory level or that a high service level is an unambiguous economic target depends crucially on the implicit non - backlogging assumption (see chapter 4). Examples for such "serious omissions" are De Groot 1970, Hey 1979, 1981, Benassy 1982 who presents the conclusion that nearly all of potential demand should be provided for.[1] The omission of making the LS assumption explicit is less serious in papers which try to develop solutions to complicated problems, where backlogging could be easily incorporated. To some extent the informed reader can at least guess that the LS case is treated, since the mathematical formulas contain a small remark, that the initial stock (before ordering), x, is assumed to be positive. (Dvoretzky 1952, Arrow 1951, Zabel 1967)[2]

1) Formulas and conclusion see column 3 resp. 4 in table 1.

2) There are papers which can be classified neither as BL or as LS cases since there are neither shortages to be backlogged nor sales lost. These papers assume that eventually unsatisfied demand is met by emergency procurement. In that case shortage is met by the individual maximizer (and not by his competitors), but shortage is not carried over into the next period. Authors who allow their shortage function - usually labelled g (.) - to be interpreted in this way and disregard the possibility of future costs stemming from a negative initial stock do not make an error, but they cannot be considered as modelling the backlog case. Whenever there is a feasibility of emergency procurement, we have two order (or production) quantities to consider (the initial optimal quantity and the ex post quantity ordered or produced). In this paper we want to differentiate between shortage costs which cannot be recovered in the next periods (this is the goodwill loss incurred since the firm could not meet demand immediately, even if the costumer finally gets an item during next period) and the revenues and costs of shortage which accrue from the later delivery (in our model these depend on the price, production cost, time discount and on the feasibility of backlogging). See appendix 1 for the problem of emergency procurement.

- The third group explicitly refers to the backlogging case and accomplishes this task in a correct way. In table 1 we classified only 4 papers in this group: Karlin, Scarf 1958, Iglehart, Karlin 1962, Scarf 1963 and Nahmias 1979. Unfortunately these authors were not interested in the special influence of backlogged orders on the actual order quantity (none of them calculated some "newsboy type" formula for the linear case with and without backlogging). These papers concentrate mainly on the conceptual problems the backlogging or the lost sales case implies for the optimality or uniqueness of the solution. Therefore the third column of table one, which aims to present an operational result for the linear case, is empty as far as correct results are concerned.

- The fourth group includes explicitly the feasibility of backlogging, but forgets about the revenues from backlogging. Backlogging has a cost, namely that the backlogged excess demand has to be produced in the next period, thus that part of the function which incorporates the expected future effects of an initial inventory is not truncated at the point $x = 0$ in the BL case, as it is in the LS case: if excess demand is backlogged, it has to be produced in the next period. This is the cost of backlogging. On the other hand it creates also revenues, namely the properly discounted revenues from selling the goods after the orders (production) are carried out. These revenues should best be incorporated in that part of the functional which is usually called the L (.) or G (.) function, which represent the one period expected losses (gains) from a certain order (production), while the f_{t-1} - or V ( ) - term contains the future consequences of an item stocked or backlogged. Iglehart, Karlin (1962) for example used this way to incorporate revenues from backlogs. We do the same in chapter 3[1]. Out of the 12 papers and textbooks which purport to treat the BL case, the revenues of backlogs are forgotten in 8 papers or books.

The crucial consequence of forgetting revenues from backlogs is that backlogging seems to be an unfavourable event and optimal production is *increased* in order to avert this cost component. On the other hand if backlogging results in revenues higher than costs ($p > c$), backlogging opens the opportunity, to delay the order or production process up to the time where uncertainty has lifted. This decreases the cost of a low inventory and optimal post order (post production) inventory stock should be lower in the BL case as compared to the LS case.

Examples of papers where the feasibility of backlogging increases optimal post order (post production) stocks are Veinott 1966 (p. 753 ff), Morton 1971 (p. 1708 ff) and Archibald 1981 (p. 175 ff). Veinott's equation 3.8. is presented in equation (1) with some modifications[2] which make it comparable to other studies. Optimal post order stock is decreasing in b, the

1) There is a slight difference between Iglehart & Karlin's modelling and mine insofar as they have an extra term regarding the case that optimal post order stock may be negative. Following the other literature we assume that post order stock - in the very contrast to pre order stock- has to be positive.

2) The following modifications were used: we assumed $c_t = c_{t+1}$ (no changes in costs from t to t + 1); Veinott's p wants to incorporate price plus goodwill costs (= p + g). If we interpreted Veinott's shortage parameter p as $[p + g - \alpha bp]$ then his model would be able to incorporate the revenues of backlogged orders correctly. This interpretation is not contradicted explicitly by Veinott's papers, but not hinted at as possible either. Moreover it would not be very practical since part of the influence of different degrees of backlogging - represented by the parameter b - would be hidden in the parameter p, while another part (see the αbc term in equation (1)) is made explicit.

parameter which represents the feasibility of backlogging (BL-case: b = 1, LS-case b = 0). Morton's equation yielding an higher optimal starting stock in the BL-case than in the LS-case are recapitulated in equation (3), it's consequence in (4). For the explanation of the variables used see appendix 2.

(1) Veinott 1966 $F(y) = \dfrac{p+g-c+\alpha bc}{p+g+h+\alpha bc-\alpha ac}$

(2) Formula (1) implies[1] $\dfrac{\partial F(y)}{\partial b} > 0$ and $\dfrac{\partial y^*}{\partial b} > 0$

y^* denotes optimal value of y

(3) Morton 1971 BL: $F(y) = \dfrac{p-c(1-\alpha)}{p+h}$

LS: $F(y) = \dfrac{p-c}{p+h-\alpha c}$

(4) Y^*_{BL} Y^*_{LS}

How did it happen that revenues form backlogs were forgotten in such excellent and seminal articles like the mentioned ones? There are several levels of answers: First, we have to concede that most of the articles were interested in solving some conceptual problems non zero lead time, interrelated demands, non linear cost etc.)

1) As mentioned above we could make an attempt to unify Veinott's formula with my formula (8), if we interprete p in a way incorporating the price (which we will labell as capital P here) and the costs of non satisfied demand: $p = (1-\alpha b)P$. Then $\partial y^*/\partial b$ would become negative. Veinott however introduced the variable p in a model without backlogs, and did not mention that its contents may have changed after incorporation of the feasibility of backlogs. And even more importent, the following literature took p at its face value (Hillier Liebermann, p. 529, Johnson Montgomery 1974, p. 53, Archibald 1981, p. 1171 ff), presenting formulas and numerical examples favouring a positive influence of b on y^*.

Table 1 A)

Survey on Inventory Models with Special Regard to the BL-case

Author	Treatment of backlogs; references to earlier papers	Result for F(y) to determine y^* in linear model	Main purpose of the analysis; interpretation of results on the impact of uncertainty on y^*
Arrow et al. 1951, p. 259	EP-interpretation with penalty assumed to be 100 times higher than c+h Fry 1928, Eisenhart 1951	---	seminal paper on inventory theory "The penalty for depleted stocks may be very high: "A horse, a horse, my kingdom for a horse, cried defeated Richard III" (p. 259)
Dvoretzky et al. 1952, p. 188, 198	implicit LS (see equ. 3.3) Masse' 1946, Arrow et al 1951	---	a "general theory" of inventory behavior (p. 187)
Bellman et al. 1956, p. 83, 85	EP-interpretation; g is a cost (p. 85) Arrow et al. 1951, Dvoretzky et al. 1952	---	application of functional equations on inventory problems
Karlin 1958, p. 136	explicit LS model, Arrow et al. 1951	---	dynamic inventory problem; in an example optimal stocks equal 331 cans while expected demand is 192
Karlin, Scarf 1958, p. 156 f	LS (=EP) and BL, correct treatment of BL	---	order delivery time lags
Mills 1959,	implicit LS	---	effect of uncertainty on optimal price $y < x$ yields a cost (goodwill loss) but $y > x$ yields a positive value of inventories
Karlin 1960, p. 232	explicit BL forgetting ERBS	LS: $\frac{p+g-c}{p+g-\alpha c+h}$ BL: $\frac{p+g-c(1-\alpha)}{p+g+h}$	demand distribution may change from time to time $y^*_{BL} > y_{LS}$
Iglehart, Karlin 1962, p. 128-137	LS and BL, correct treatment of BL	---	optimal policy for correlated demand distributions
Karlin, Carr 1962, p. 162	implicit LS	---	joint optimization of prices and quantities the interpretation of the results concentrate on prices (under multiplicative vs. additive uncertainty)

LS ... Lost Sales Model
BL ... Backlog Model
EP ... Emergency Procurement
ERBS ... Expected Revenue of Backlogged Sales

Table 1 B)

Author	Treatment of backlogs references to earlier papers	Result for $F(y)$ to determine y^* in linear model	Main purpose of the analysis interpretation of results on the impact of uncertainty on y^*
Mills 1962, p. 117 ff	purports that shortage function allows alternatively LS, EP or BL interpretation (p. 108, 118); if BL is used then ERBS are forgotten	$\frac{p+g-c}{p+g+h-c}$	multiperiod price and output decision for constant price probability of shortage lower than in one-period model (on p. 119, 1:25); shortage is a cost, inventories a source of revenue
Scarf 1963, p. 199	explicit BL, L(y) not specified to see if ERBS are forgotten or not (p. 199); Arrow et al. 1951, 1958	---	survey an inventory policy that permits large shortages to occur persistently will generally be of little value (p. 189)
Veinott 1963, p. 90-112	explicit BL and LS cases forgetting ERBS for BL case Karlin 1960	---	critical numbers for BL and LS
Veinott 1966, p. 753-755	explicit BL forgetting ERBS	$\frac{p+g+\alpha bc-c}{p+g+\alpha bc+h-\alpha ac}$	survey on inventory literature no interpretation, but the formula 3.8 yields optimal stock increasing with the feasibility of backlogging (b)
Zabel 1967, p. 196	implicit LS Mills 1962	---	price behavior of competitive firms
De Groot 1970, p. 409 f	implicit LS Arrow et al. 1951 Dvoretzky et al. 1952	$\frac{p-c}{p-\alpha c}$	infinite stages inventory problem no great loss by having stocks unsold even for a long period; implicity for $\alpha \to 1$, $y^* \to \infty$
Zabel 1970, p. 205	implicit LS (I = non negative)	---	price behavior of monopolies
Morton 1971, p. 1708-1712	explicit BL and LS forgetting ERBS	BL: $\frac{p-c(1-\alpha)}{p+h}$ LS: $\frac{p-c}{p+h-\alpha c}$	myopic rules as approximations to costly calculations $y^*_{BL} > y^*_{LS}$; in the text on the mixed case Morton distinghishes between p and P (p. 1715), which would allow a correct solution, but no P is to be found in the respective formula
Zabel 1971, p. 122	implicit LS Zabel 1967	---	impact of risk attitude on competitive firm
Zabel 1972, p. 524	explicit LS	---	optimal policy with multiplicative vs. addition uncertainty

Table 1 C)

Author	Treatment of backlogs references to earlier papers	Result for F(y) to determine y^+ in linear model	Main purpose of the analysis interpretation of results on the impact of uncertainty on y^*
Johnson, Montgomery 1974, p. 53	explizit BL forgetting ERBS	$\frac{p-c-\alpha c}{p+h}$	quantitatively oriented textbook in an example $y^* > E(x)$
Hey 1979, p. 160 ff	implicit LS De Groot 1970	$\frac{p-c}{p-\alpha c}$	textbook on uncertainty for $\alpha \to 1$, Y^* approaches the maximum value that x may take (p. 161)
Nahmias 1979, p. 917	explicit (partial) BL Morton 1971, Karlin, Scarf 1958	---	workable solutions to realistic problems defines different shortage costs for LS and BL (p. 917), so that the second one may be interpreted to incorporate ERBS
Hillier, Liebermann 1980, p. 529	explicit BL forgetting ERBS	$\frac{p-c(1-\alpha)}{p+h}$	textbook since α near to unity (p. 531 0,995) a very high proportion of potential demand (92,7 %) should be provided for
Archibald 1981, p. 1171-1177	explicit BL forgetting ERBS Hadley, Whitin 1963 Morton 1971	---	costs of approximations optimal inventory in BL larger than in LS (f.e. 20 % on p. 1176)
Hey 1981, p. 135	implicit LS	$\frac{p-c}{p-\alpha c+\alpha h}$	textbook
Arran, Moses 1982, p. 190 A	implicit LS Arrow et al. 1958/1962 Dvoretzky et al. 1952 Hadley, Whitin 1963	---	joint determination of prices and inventories "dynamic strategy may lead to an overprovision" (p. 192)
Benassy 1982,	implicit LS Arrow et al. 1958 Bellman 1956	$\frac{p-c}{p-\alpha ac}$	optimal policy under uncertainty

or in finding a cheap approximation algorithm (Archibald 1981) and not in the particular influence of backlogs on optimal inventory. The main purpose of the papers are listed in column 4 of table 1 to demonstrate this contention. Secondly the literature stated initially that expected profits depend on price times demand or price times production, whatever is smaller, and then that the optimal order did not depend on demand (since demand does not matter if we differentiate equation (5) with respect to production). The "rule" that "demand does not matter" is stated extremely well in Veinott 1962 (p. 96) but it is true only for the one period decision or the LS case. Demand matters however in the disguise of backlogged excess demand, (see p . g for differentiating the 2^{nd} term in the second line of equation 7).

$$(5) \qquad E\pi = p.\min(x,q) - c\,q$$

$$(6) \qquad \frac{\partial E\pi}{\partial q} = p \int_q^{\infty} f(x)dx - c$$

Thirdly, in many papers we cannot really be sure whether the authors treated the backlog issue only in a somewhat loose manner or whether they actually made an error. Take for example the Veinott or the Morton papers. Both did not specify the penalty function in a strict way. The penalty term ($g^{Veinott}$ or g^{Morton}) could in each of the papers contain the value of future revenues of backlogged demand

($g^{Morton} = g-p\ (1-\alpha))$, though this is not hinted at. However, in that case g^{Morton} should be different in the BL-case, $(g-p\ (1-\alpha))$, and in the LS-case, (g), which should have been mentioned.[1] The further literature took the results at their face value and did not notice that the backlog case yielded higher optimal stocks than the lost sales case (e.g. Archibald 1981).

We do not want to be picky on the question whether the mentioned treatment of revenues from backlogs should be called a looseness or an error, but rather present the facts: Veinotts formula is correct if b is partly hidden in $g^{Veinott}$ partly explicit in the other part of the formula, Morton is correct if g^{Morton} is $g-p\ (1-\alpha)$ in the BL case and g in the LS case. Both papers did not aim at the correct incorporation of backlogs and are too important to be judged at this issue.

Finally inventory literature heavily depends on some central publications: in the fifties every paper was based on Arrow et al 1951, Dvoretzky et al 1952, in the sixties on the two Arrow books (1958, 1962) and then on the excellent papers of Veinott. The second column of table one demonstrates this interrelatedness of the literature, which helped to maintain an error so persistently.

1) In Veinott's paper $g^{Veinott}$ could be interpreted at $g^{Veinott} = p + g - \alpha bp$ with $b = 0$ in the LS and $b = 1$ in the BL case, now part of the influence of the backlogging feasibility is hidden in $g^{Veinott}$, part is made explicit in his formula 3.8 (our equation 1).

3. The Correct Treatment of Backlogs

The correct treatment of backlogs is demonstrated in the literature in the papers of Karlin, Scarf 1958, Iglehart, Karlin 1962, mainly for complicated models.
Most of their features, and especially the effect of backlogging can be demonstrated by means of a simple linear cost model, with zero lead time and a known and (from period to period) independent demand probability function, f (x). For the infinite horizon we apply the usual dynamic programming technique to obtain recursion (7) which constitutes an optimal strategy for the post order - (post production) stock, y, as seen from the position of an initial (pre order) inventory, I. (y-I=q, the order or production quantity). The first term in each curled bracket represents the sales revenue as in the one period model. The h (.) and g (.) terms represent holding and goodwill costs, the argument in both cases is the difference between the post orderstock (y=I+q) and the demand. The V (.) terms calculate the properly discounted future expected consequences from an item stocked or a demand backlogged. In the first case the consequence is positive since an item stocked decreases future production costs (depending on the discount parameter, α, and the durability parameter a). In the second case it is a cost since backlogged demand has to be produced next period (depending on the degree of backlogging b). The second term in the second line is the term often forgotten in the literature, it represents the expected revenue of backlogged demand (ERBS).

$$(7)\quad V(I) = \max_{y-I \geq 0} \int_0^y \left\{ [px - h(y-x)] + \alpha V[a(-x+y)] \right\} f(x)dx +$$

$$+ \int_y^\infty \left\{ py + \alpha bp(x-y) - g(x-y) + \alpha V[b(-x+y)] \right\} f(x)dx - c(y-I)$$

Using dynamic programming techniques the solution for the optimal post order stock is given by (8).

$$(8)\quad BL:\ F(y) = \frac{(p-c)(1-\alpha b) + g}{p+g+h-\alpha b(p-c)-\alpha ac} \qquad 0 < b \leq 1$$

$$(9)\quad LS:\ F(y) = \frac{p-c+g}{p+g+h-\alpha ac} \qquad b = 0$$

$Y^*_{BL} < Y^*_{LS}$ and in general (8) implies $\frac{\partial F(y)}{\partial b} < 0$

and $\frac{\partial y^*}{\partial b} < 0$

If backlogs are not feasible (b = 0) we get the result of equation 9, in general the optimal production (starting stock) decreases in b. This is the contrary result to Veinott 1966, Johnson, Montgomery 1964, Archibald 1981. The result of these papers can be derived if the second term in the second line of 7 is omitted.

Table 2

Overall Record of 27 Inventory Models

LS - Models		BL - Models		Emergency Procurement Models 3)
explicit LS	implicit LS 1)	correct incorporation of ERBS 2)	forgetting ERBS	
Karlin 1958	Dvoretky et al. 1952	Karlin, Scarf 1958	Karlin 1960	Arrow et al. 1951
Zabel 1972	Mills 1959	Iglehart, Karlin 1962	Mills 1962	Bellman 1952
	Karlin, Carr 1962	Scarf 1963	Veinott 1963	
	Zabel 1970	Nahmias 1979	Veinott 1966	
	De Groot 1970		Johnson, Montgomery 1974	
	Hey 1979		Hillier, Lieberman 1980	
	Hey 1981		Archibald 1981	
	Benassy 1982			
	Arran, Moses 1982			
	Zabel 1967			
	Zabel 1971			
2	11	4	8	2

1) Including the cases where the non negativity constraint on the initial inventory is given in the mathematical part

2) Including the cases where different shortage function are defined for the BL and LS case, which are not specified so that they could incorporate the ERBS (Nahmias 1979)

3) Including the cases where emergency procurement costs are defined to be positive unambigously

4. Implications of Backlogging on the Optimal Inventory

In the last chapter we demonstrated that the correct incorporation of backlogs yields the result that backlogging decreases optimal inventory. This is a matter of logic as far as we can see.
There are several other implications of the formula (8) for the optimal starting stock for a producing firm, which depend on empirical parameters.
The first is, that it seems very likely - as seen from formula (8)-that the optimal starting stock is lower than expected demand and not quite a bit above it -(as assumed by "service levels" of 90 % or more[1]), given that backlogging is a real world phenomenon e.g. in an industrialized country like that US. Let us assume complete backlogging and complete durability of goods ($a = b = 1$), goodwill cost and holding costs are pretty small ($g = h = 0,1$), we standardize the prize of the good to be 1, its production costs as 0,8, let the discount parameter α be 0,9. This yields that we should choose y in a way to get $F(y) = 0,4$, this means a post production stock less than expected demand.

1) See for example Arrow et al (1951), many contributions in the books of Arrow et al 1958, 1962 and other examples in literature, where "service levels" are assumed to be 90 % or more.

The result depends crucially on the assumption that costs are the major part of revenue (profit is the minor one)[1], which seems realistic for a roughly competitive economy. Therefor net inventory after production and after demand occurred (the initial stock of the next period) should be negative on average. Another implication is, that if it is true that inventories (starting stocks) on average are negative, then an increase in the degree of uncertainty (in the sense of a mean preserving spread) should result in a decrease of the optimal starting stock (of production) in the sense of making it even more negative[2]. Contrary to the often implicitely or explicitely stated assumption that demand uncertainty increases inventories (even constituting one of its main origins) in fix price models, uncertainty would yield lower finished goods inventories

1) This will be the case even if we compare with profits the variable costs only (which is the only correct way as was mentioned in a comment by Peter Swoboda).
Models where q is interpreted as production (and c therefor as production cost) are more likely to call for a low inventory because production costs are high in contrast to the costs of ordering (in models for optimal ordering policy).

2) Let us assume two symmetric density functions g (x) and f(x), with distribution functions G (x) and F(x) respectively. Both density function have the same expected value, but g (x) has more "weight in the tails". The difference of the distribution functions change their sign only once ("mean preserving spread" according to Lippmann, McCall 1981, p. 215), namely at $E(f(x)) = E(g(x))$. For $F(Y) < \frac{1}{2}$, $F(Y)$ is smaller than $G(Y)$, for $F(Y) > \frac{1}{2}$ the contrary is true. It follows that if there is excess demand on average, then increasing the degree of uncertainty decreases optimal stock.

or a higher order backlog. If backlogging is feasible, firms will try to delay their production decision up to the time where demand is known. Finally it seems to me that the shift from standardized basic goods (steel, paper, raw materials) to the more sophisticated and specialized durable goods which sometimes have to be modified for each and any customer (e.g. industrial plants), may result in increasing the feasibility of backlogging and decreasing the importance of stocks of finished goods. If this hypothesis is true, net inventories should exhibit a secular decreasing trend (in the sense of becoming more negative).

5. Empirical data on net inventories

Testing these hypotheses (negative correlation with uncertainty, structural shifts) is not an easy thing. The term "net inventories" is commonly used in the Management Science Literature. There seems to be agreement that on the theoretical level net inventories should be defined as the difference between the order backlog and the (finished goods) inventories. But there is little evidence (even little discussion) whether in general these net inventories are negative or positive, whether this is true for some industries while the contrary is true for others. One of the few exceptions is the paper by Badinelli and Schwarz (1984), which shows that optimal inventories should be high at the retailers and correspondingly low (in the sense of a negative safety stock) at the warehouse. If there were more evidence on the empirical level it should be easy to discriminate between the contention of many papers that the service level should be near 100 % (implying a large positive inventory level on average) on the one hand and the implications of chapter 4 in this paper, that the empirical cost parameters would in general yield negative net inventories.

The lack of empirical evidence is due to the difficulty of translating the theoretical concept of net inventories into available empirical data. The only concept used so far on the aggregate level (e.g. for total manufacturing) is to define net inventories as the difference of finished goods inventory stock and unfilled orders. This concept was applied on the micro-level in the seminal paper by Holt et al 1960 (p. 315, "net inventory i.e. . gross inventory minus unfilled orders ...") on the macro level by Child (1967)[1].
Defining net inventies as the difference between finished good and the order backlog is not at all an innocent concept[2]. The order backlog may well be partly due to technical reasons, the aggregate over all firms will constitute a mixture of firms which are used to produce on order and of those which produce on stock. The reasons for holding a backlog and an inventory as well as their costs will be different[1]. But if service levels near 90 % or 100 % were "natural" finished goods should be at least as large as the order backlog, even if the order stock as measured yield a several overestimation of backlogged demand.

1) Zarnowitz 1973 prefers the stock/orders relationships (chapter 344 ff) instead of a net concept Amihud and Mendelson (1983) may be cited as a recent microeconomic study defining net inventories as inventories minus unfilled orders.

2) A summary of objections see in Rowley and Trivedi (1975) or Zarnowitz (1973).

If we nevertheless tentatively accept that net inventories could be approximated by the difference[1] between finished goods inventories and backlogged orders, net inventories are unaequivocally negative in modern industrialized countries.
In the US finished goods stocks amount to 14,2 % of annual sales (1954-1982, Report of the President, see table 3), order backlogs to 28,9 %, net inventories are therefore negative (-14,7 % of sales)[2]. In Germany (1980-1983) finished stocks amount to 7,5 %, order backlog to 23,9 %, the net balance is -16,2 %. In Austria, 1980/82, inventories and backlogs amount to 8,3 % and 25,8 % respectively, the net position is negative (-17,5 %). With all provisions for the poor quality of the data and for aggregating over industries, we can conclude that for the aggregate of manufacturing net inventories are negative in industrialized countries.
Maybe the first years of the eighties (up to the start of the recent upswing) could be regarded as time of increased demand uncertainty. The negative inventory position reached -17,3 %

1) Further limitations for the results arise from the fact that we have to use data for inventories and backlogged orders which are not strictly comparable, we have to accept their difference "finished goods inventory minus order backlog" as the net inventory and we realize that the beginning of the eighties were not only a period of increased uncertainty (but also of rising interest rates, divergent price expectations and shifting demand expectations).

2) This calculation confronts the relative inventory position of all firms, with the order backlogs of the firms which usually do produce at least partially on order. The net inventories would still be negative if the absolute volume orders of this group would be confronted with the total sales of all firms.

Finished Goods Inventories and Order Backlog in % of Annual Sales

U S A	(1) Inventory - Sales-Ratio	(2) Order-Shippment-Ratio[1] All	(3) Order-Shippment-Ratio[1] Durables	(4) Order-Shippment-Ratio[1] Non Durables	Net-inventories (1)-(2) = (5)
Ø 54-82	14,22	28,90	34,49	6,53	- 14,68
1954-59	14,58	28,29	33,42	8,02	- 13,71
1960-69	14,17	27,03	32,40	6,09	- 12,86
1970-79	14,09	30,38	36,20	6,53	- 16,29
1980-82	14,08	31,39	38,08	5,03	- 17,31
1983	12,81	29,42	35,75	4,67	- 17,42

Source: Economic Report of the President (1984).

1) For firms which do have backlogs
Unfilled orders in % of sales (including branches without backlogs): 16,43
Net inventories: -2,75 (13,68 - 16,43) (1980-1983).

Germany	(1) Inventory - Sales-Ratio	(2) Evaluation as too high (+) as too low (-)	(3) Order backlog	(4) Evaluation as too high (+) as too low (-)	(5) Net - inventories (3)-(1)
1972	-	+ 14	23,5	- 26	-
1980-82	7,5	+ 21	23,9	- 40	- 16,4
1983	7,9	+ 17	21,7	- 43	- 13,8

Source: IFO Munich

Austria[1]	(1) Inventory - Sales-Ratio	(2) Evaluation as too high (+) as too low (-)	(3) Order backlog	(4) Evaluation as too high (+) as too low (-)	(5) Net - inventories
Ø 73/74	7,8	+ 9	22,9	- 4	- 15,1
Ø 80/82	8,3	+ 26	25,8	- 26	- 17,5
1983	7,3	+ 25	24,9	- 40	- 17,6

1) Excluding maschinery and those branches, not covered by order statistic

in the US (1980/82), after -16,3 % in the decade before. In Austria it decreased to -17,5 %, after -15,1 % in the decade before. The negativity arose (most surprisingly from the point of view of the trough at the start of the eighties, but reasonable from the view of optimal inventory behavior) from an increased order stock. Firms were reluctant to produce if there was a chance to wait until demand uncertainty had lifted.

There is of course a secular shift in the net position. In the fifties (1954-1959) the net position was -13,7 %, 1983 it amounted to -17,4 % in the US. In Austria the net inventories amounted to -15,1 % in 1973/74, but -17,6 % in 1983.

We want to repeat the limitations of the evidence presented, especially that the measured order backlog may severely overestimate the backlogged demand, since it incorporates technically necessary backlogs. But at least we want to conclude that there is no evidence that positive net inventories are a "natural" fact and that more evidence - starting from the micro level - should be gathered to determine the net inventory position. It seems worth gathering material on under which conditions for which branches and situations negative net inventories are economically reasonable.

6. Summary

This paper contends that most articles and books do not assess the role of backlogs correctly. Partly they refer explicitly to the LS case (2 out of 27 publications classified in table 1),this is an analytically clean way, but it omits an important real world strategy. Partly the literature refers implicitly to the LS case (11 cases), this is an omission. Another part of the literature refers to the BL case but forgets the revenues of backlogged orders (8 cases), resulting in the implausible result that the feasibility of backlogs increases optimal stocks. The minority (4 cases), which treats backlogs correctly, was interested in the solutions of problems, but not in the more prosaic question of the potential influence of backlogs on the optimal post order (post production) inventory position. If we correctly include the influence of revenues from backlogged orders, the feasibility of backlogging decreases optimal stocks. This result is easy to demonstrate (this is done for a simple infinite horizon model with i.i.d. demand) and should be acknowledged even by readers who do not like to follow the further results.

A very simple linear model shows that it is very likely that net inventories should be negative in a manufacturing industry and that-if this is the case-then increasing demand uncertainty should further decrease optimal inventory. The feasibility of backlogging should increase in an economy whose structure changes from basic goods to sophisticated specialized goods. Empirical data are presented which show that these three hypotheses are at least not grossly at variance with empirical facts. Further empirical investigation on the optimality of negative net inventories at least for certain industries seems to be necessary.

Appendix 1: The No-Backlog and No-Lost Sales Case: Emergency Procurement (EP Case)

We have distinghished between the alternatives of the Backlog Case (BL) and the Lost Sales Case (LS). If shortages are met by emergency procurement (emergency order or overtime production), there are neither lost sales nor backlogs (Bellman et al 1956, Mills 1962). The costs of shortages consist of higher ordering (production) costs as compared to "normal" ordering (production) costs.

The costs of emergency procurement are included in the shortage function - usually labelled as G (.) or L (.) function. Often the authors allow this function to be interpreted either as referring to goodwill losses due to the effect that the firms may aquire a reputation of being unreliable as a supplier or alternatively as the costs of emergency procurement.

If this interpretation of the G (.) function is given the results of the LS case and of the EP case are formally identical.

However there is an important difference in the interpretation of the shortage function in the two cases and to my opinion there are also quite different empirical implications.

Shortage costs in the sense of goodwill losses are unambigously costs (negative revenues). Emergency procurement on the other side may constitute a net profit: if the emergency procurement costs are less than the selling price of the good, EP yields a net profit component. This is not seen from the literature, which interprets the shortage function as a cost function without allowing that these costs may be negative since revenues are higher than costs. This may stem from the comparison of the case where demand is met by normal production (order) and the case where it is met by emergency production. Of course the second case is more costly (otherwise there would be no normal production) or in other words profits per unit are smaller in the case of emergency procurement. Most realistically they will be very near to zero. They may even be negative if emergency procurement costs are larger than sales revenue (the firm will do this if it is forced legally or by some long run consideration to supply the good even if this infers a loss). But in general the shortage function may constitute a source of profits in the expected profit maximization procedure in the EP case, albeit a smaller one than if the good had been provided by normal production. This is forgotten by authors who implicitly or explicitly refer only to costs of emergency procurement.

If g turns from beeing negative (goodwill loss as a cost) to zero (if emergency procurement costs equal sales price) or negative (if there is a small but positive profit from this source) the optimal (preliminary) starting stock (production) decreases.

$$(9) \quad \text{LS case} \quad F(y) = \frac{p-c+g}{p+g+h-\alpha ac}$$

Equation (9) - the LS-equation - is formally identical with the EP case. But now interpret g as $[c_{emerg} - p]$ [1), where the emergency procurement costs are higher than the cost of normal ordering or production ($c_{emerg} > c$). Then (9) turns into (9 A)

$$(9\ A) \quad \text{EP case} \quad F(y) = \frac{p-c+\overbrace{(c_{emerg}-p)}^{g}}{p+\underbrace{(c_{emerg}-p)}_{g}+h-\alpha ac}$$

1) The form $[c_{emerg}-p]$ instead of $[p-c_{emerg}]$ is necessary since the shortage function G (.) is deducted from the expected profits (see equation 7). If it had been defined as a revenue in (7) then g would get a negative sign in (9) and we could interpret it as $g = p-c_{emerg}$.
We assume the numerator of 9 A to be positive which is somewhat stronger than the usual assumption that p-c should be positive.

Empirical fact decide whether g is positive or negative. If emergency procurement costs are larger than the product price, but emergency procurement is done nevertheless, then optimal production is larger in the EP case as compared with the LS case[1]. If emergency costs are smaller than p, optimal production is smaller (g is negative). However we do not expect g to be very large and especially do not allow it to be 100 times as large as shortage costs (Arrow et al. 1951, p. 259). This asymmetry can be justified only for military purposes of if we forget that g is a net cost (consisting of a revenue and a cost component).

1) To make things comparable we assume g for th LS case to be zero. We could also say alternatively that the EP case yields a higher value of y^* as long as the net difference of the costs of emergency procurement and the price are larger than g in the LS case.

Appendix 2: List of Assumptions and Abbreviations

Assumptions: infinite horizon
continuous time
profit maximization
linear costs
fixed constant price
demand uncertainty with known probability function (i.i.d.)
no delivery lag (orders or production has to be decided upon before actual demand is known)
no emergency procurements

I initial inventory (inherited from past, before ordering or production)

q order quantity or production quantity

y=q+I post order (production) stock, before demand (starting stock)

x,f(x), F(x) demand, probability density of demand, distribution function

p, c price, production (ordering) cost per unit

h, g cost of holding 1 unit of stock respectively goodwill loss for 1 unit of unsatisfied demand. The goodwill loss cannot be recovered even in the complete BL case, it is the present value of losses accruing from the reputation of being an unreliable supplier even if the costumer sticks to the supplier for today's unsatisfied demand.

a, b durability parameter (a=1 for durable goods, a=0 for perishable goods) resp. backlogging parameter (b=1 if no demand is lost, b=0 if all unsatisfied demand is lost)

V (I) is the maximized total expected discounted profits as viewed from a moment, where the initial inventory is I

G (.) function describing the (one period) expected shortage and inventory holding cost. It consists of a h (.) and a g (.) part and includes in case of backlogging also the discounted value of future revenues of backlogged demand (though these are not "one-period" costs in the usual sense)

$\alpha = \frac{1}{1+r}$ discount factor (r = discount rate)

LS, BL, EP lost sales case, backlog case, emergency procurement case

ERBS expected revenue of backlogged sales

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